# HIGHER SECANT VARIETIES OF THE MINIMAL ADJOINT ORBIT

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Abstract

The adjoint group of a simple complex Lie algebra $\mathfrak{g}$ has a unique minimal orbit in the projective space $\mathbb{P}\mathfrak{g}$, whose pre-image in $\mathfrak{g}$ we denote by $C$. We explicitly describe, for every classical $\mathfrak{g}$ and every natural number $k$, the Zariski closure $\overline{kC}$ of the union $kC$ of all spaces spanned by $k$ points on $C$. The image of this set in $\mathbb{P}\mathfrak{g}$ is usually called the $(k-1)$-st *secant variety* of $\mathbb{P}C$, and its dimension and defect are easily determined from our explicit description. In particular, it follows that the smallest $k$ for which $\overline{kC}$ is equal to $\mathfrak{g}$, is $n$ for $\mathfrak{sl}_n$, $2n$ for $\mathfrak{sp}_{2n}$, 4 for $\mathfrak{o}_7$, and $\lfloor \frac{n}{2} \rfloor$ for $\mathfrak{o}_n$, $n \geq 8$; we find that the upper bound on this $k$ provided by a theorem of Zak on secants of general varieties, is off by a factor of 2 in the cases of $\mathfrak{sl}_n$ and $\mathfrak{o}_n$, but sharp for $\mathfrak{sp}_{2n}$.

The orthogonal Lie algebras turn out to be the most difficult, by far: while all sets $kC$ are closed in the other two cases, this is not true for $2C$ in $\mathfrak{o}_n$, and we discuss the problems arising in describing the sets $kC$ for $k \geq 3$. In particular, we do not know the smallest $k$ for which $kC$ is equal to $\mathfrak{o}_n$, though we do prove that it is at most $\lfloor \frac{n}{2} \rfloor + 3$.


## 1. Introduction and results

The projective space $\mathbb{P}\mathfrak{g}$, where $\mathfrak{g}$ is a semisimple Lie algebra over an algebraically closed field $K$ of characteristic zero, has a unique minimal orbit under the action of the adjoint group of $\mathfrak{g}$; let $C$ denote the pre-image of this orbit in $\mathfrak{g}$. Here 'minimal' refers to the inclusion order among orbit closures, so the minimality of $\mathbb{P}C$ means that it is contained in the closure of any other orbit. The set $C$, itself a nilpotent orbit, plays an important role in several branches of Lie theory: First, $C$ consists of all long root vectors relative to appropriate Cartan subalgebras (or of all highest root vectors relative to Borel subalgebras) and is therefore of interest in representation theory. Alternatively, $C$ may be described as the set of all non-zero $X \in \mathfrak{g}$ for which $[X,[X,\mathfrak{g}]] \subseteq KX$ [13], and these *extremal elements* pop up in the classification of Lie algebras in positive characteristic [6, 19] (for a possible connection between our results and those of [6], see the conclusion of this paper). We, now, are to discuss properties of $C$ that are interesting from a geometric point of view, namely: what do the higher secant varieties of $\mathbb{P}C$ in $\mathbb{P}\mathfrak{g}$ look like, and what are the corresponding defects of $\mathbb{P}C$? This work is part of a larger project, which asks for the secant varieties of the minimal orbit in any irreducible representation of any reductive algebraic group.

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Recall that the $(k-1)$-st *secant variety* of $\mathbb{P}C$ in the projective space $\mathbb{P}\mathfrak{g}$ is the Zariski closure of the union of all projective subspaces of $\mathbb{P}\mathfrak{g}$ spanned by $k$ points on $\mathbb{P}C$. As $C$ is a cone, this secant variety is in fact equal to $\mathbb{P}(\overline{kC}\setminus\{0\}) = \overline{\mathbb{P}(kC\setminus\{0\})}$ where $kC$ is the set defined by

$$kC := \{J_1 + \ldots + J_k \mid J_i \in C \text{ for all } i\}.$$

The *expected dimension* of $kC$ is $\min\{k \dim C, \dim \mathfrak{g}\}$, and this expected dimension minus the actual dimension of $kC$ is usually called the $(k-1)$-*defect* of $C$ (or of $\mathbb{P}C$).

We are to present explicit descriptions of the sets $\overline{kC}$ in the case where $\mathfrak{g}$ is classical, which extend the results on the *first* secant variety of $\mathbb{P}C$ by Kaji *et al* [13, 14]. It should be mentioned that their method applies to the exceptional simple Lie algebras, as well, while it is not obvious how to uniformise our case-by-case approach so as to incorporate those in our treatment. Closely related to the matter of this paper is also [4], which treats the higher secant varieties of the variety of indecomposable tensors in a tensor product.

The research on higher secant varieties of general varieties finds its origin in the works of Palatini and Terracini [18, 20], and an important part of this research concerns (bounds on) the dimensions of secant varieties, as well as the construction of concrete varieties attaining these dimensions [1, 2, 3, 8, 10, 17, 21]. The monograph [21] by Zak contains the following result that we compared to our concrete situation: suppose that the first defect $\delta$ of an irreducible smooth $n$-dimensional projective algebraic variety $X$, lying in and spanning $\mathbb{P}^N$, is non-zero. Then the smallest $k$ for which the $k$-th secant variety of $X$ is equal to $\mathbb{P}^N$, is at most $\lfloor \frac{n}{\delta} \rfloor$. Though derived for application to varieties that—unlike the minimal orbit $\mathbb{P}C$—have low codimension in the ambient projective space, this bound turns out to be quite good for the minimal orbit: it is roughly twice the actual value for $\mathfrak{sl}_n$ and $\mathfrak{o}_n$, and it is sharp for $\mathfrak{sp}_{2n}$.

Turning our attention to a classical Lie algebra $\mathfrak{g}$, we define the *rank* of an element $A \in \mathfrak{g}$, denoted $\operatorname{rk}(A)$, to be its rank as a linear map in the standard $\mathfrak{g}$-module $V$. In the cases of $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$ the minimal orbit $C$ consists of all elements of rank 1, and the following theorem, the subject of Section 2, identifies the secant varieties of study as certain determinantal varieties.

**Theorem 1.1.** *If $\mathfrak{g} = \mathfrak{sl}_n$ $(n \geq 2)$ or $\mathfrak{g} = \mathfrak{sp}_{2n}$ $(n \geq 2)$, then we have $kC = \overline{kC} = \{A \in \mathfrak{g} \mid \operatorname{rk}(A) \leq k\}$ for all $k$, $2 \leq k \leq n$.*

This is not new for $\mathfrak{sl}_n$: [15] contains a proof using the Jordan normal form, while our proof was inspired by [12, §56, Exercise 6].

**Corollary 1.2.** (1) *For $\mathfrak{g} = \mathfrak{sl}_n$ and $1 \leq k \leq n$ the dimension of $kC$ is $2kn - k^2 - 1$, so that the $(k-1)$-defect of $C$ is $\min\{(k-1)^2, (n-k)^2\}$.*
(2) *For $\mathfrak{g} = \mathfrak{sp}_{2n}$ and $1 \leq k \leq 2n$ the dimension of $kC$ is $\binom{2n+1}{2} - \binom{2n+1-k}{2}$, so that the $(k-1)$-defect of $C$ is $\min\{\binom{k}{2}, \binom{2n+1-k}{2}\}$.*

The result for $\mathfrak{o}_n$ is radically different; we assume $n \geq 7$ here, as the other (simple) cases are dealt with by the preceding theorem. Now we have

$$C = \{J \in \mathfrak{o}_n \mid \operatorname{rk}(J) = 2 \text{ and } J^2 = 0\} \text{ (see Subsection 3.1)},$$

and one might hope that $kC$ is simply the set of all elements of rank at most $2k$—but this is not true! To describe the first and the second secant variety, denote by $S_2 \subseteq$

$\mathfrak{o}_n$ the set of all semisimple elements of rank 4 whose non-zero eigenvalues (on $V$) are $a, a, -a, -a$ for some $a \in K^*$. Similarly, let $S_3 \subset \mathfrak{o}_n$ be the set of all semisimple elements of $\mathfrak{o}_n$ of rank 6 with 6 distinct non-zero eigenvalues $a, b, c, -a, -b, -c \in K$ satisfying $a + b + c = 0$. We then have the following theorem (Section 3).

**Theorem 1.3.** *For $\mathfrak{g} = \mathfrak{o}_n$ $(n \geq 7)$ the sets $\overline{2C}$ and $\overline{3C}$ are equal to $\overline{S_2}$ and $\overline{S_3}$, respectively, while for $k \geq 4$ we have $\overline{kC} = \{A \in \mathfrak{o}_n \mid \mathrm{rk}(A) \leq 2k\}$.*

**Corollary 1.4.** *For $\mathfrak{g} = \mathfrak{o}_n$ $(n \geq 7)$ the dimensions of $C, 2C, 3C$, and $kC$ $(4 \leq k \leq \lfloor \frac{n}{2} \rfloor)$ are $2n-6, 4n-13, 6n-22$, and $\binom{n}{2} - \binom{n-2k}{2}$, respectively (the dimension of $4C$ in $\mathfrak{o}_7$ is $\binom{7}{2}$). Hence, the $(k-1)$-defect of $C$ is equal to $1$ for $k = 2$, equal to $4$ for $k = 3$, and equal to $\min\{k(2k-5), \binom{n-2k}{2}\}$ if $k \geq 4$ (and zero for $(k, n) = (4, 7)$).*

Note that Theorem 1.3 only mentions the closures $\overline{kC}$, not the sets $kC$ themselves. This is because we do not know their exact structure; we now list what we do know. First, the set $2C$ is already not closed; indeed, Kaji *et al* determined the nilpotent orbits lying in its closure [14], and it turns out that $\overline{2C} \setminus 2C$ consists of a single such orbit. To formulate our proposition to that effect, recall that nilpotent orbits of $\mathrm{O}_n$ on $\mathfrak{o}_n$ correspond, through the Jordan normal form, to partitions of $n$ whose even entries have even multiplicities. If $\mathbf{d} = (d_1, \ldots, d_m)$, $d_1 \geq d_2 \geq \ldots \geq d_m$ is such a partition, then we denote by $\mathcal{O}[\mathbf{d}] = \mathcal{O}[d_1, \ldots, d_m]$ the corresponding nilpotent orbit. For example, in this notation we have $C = \mathcal{O}[2, 2, 1^{n-4}]$.

**Proposition 1.5.** *The set $2C$ is equal to $\overline{2C} \setminus \mathcal{O}[3, 2, 2, 1^{n-7}]$.*

The fact that $\overline{2C} \setminus 2C$ is a nilpotent orbit suggests to determine, for a general nilpotent orbit $\mathcal{O}$, the smallest $k$ for which $\mathcal{O}$ is contained in $kC$. Our partial result in this direction uses the notation $l(\mathbf{d}) := |\{i \mid d_i \text{ is odd, } d_i > 1\}|$. Furthermore, by the *rank* of an orbit $\mathcal{O}$ we shall mean the rank of an element of that orbit.

**Theorem 1.6.** *Let $\mathbf{d}$ be a partition of $n$ as above, and let $2k$ be the rank of $\mathcal{O}[\mathbf{d}]$. Then $\mathcal{O}[\mathbf{d}]$ is contained in $(k+1)C$. If moreover $l(\mathbf{d})$ is even, or if $l(\mathbf{d})$ is odd and $d_1 > 5$, then $\mathcal{O}(\mathbf{d})$ is already contained in $kC$.*

The upper bound $k + 1$ (notation as in the preceding theorem) is sharp for $\mathbf{d} = [3, 2, 2, 1^{n-7}], [3, 1^{n-3}]$, and $[5, 1^{n-5}]$. Hence, the nilpotent orbits of smallest rank for which we do not know the smallest $kC$ containing them, are $\mathcal{O}[3, 3, 3, 1^{n-9}]$ and $\mathcal{O}[5, 2, 2, 1^{n-9}]$, both of rank 6. In conclusion, it seems hard to write a general element of $\mathfrak{o}_n$ as a sum of as few as possible elements of $C$. The rank reduction argument used to prove Theorem 1.3, however, does give an upper bound to the maximum number of terms needed.

**Theorem 1.7.** *Every element of $\mathfrak{o}_n$ having rank at most $2k$ lies in $(k+3)C$. In particular, $(\lfloor \frac{n}{2} \rfloor + 3)C = \mathfrak{o}_n$.*

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## 2. Appetisers: $\mathfrak{sl}_n$ and $\mathfrak{sp}_n$

For all classical simple Lie algebras $\mathfrak{g}$ the minimal orbit $C$ consists of matrices of some low rank $r$ ($r = 1$ for $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$, and $r = 2$ for $\mathfrak{o}_n$; see below). As a result, an element of $kC$ has rank at most $kr$. Now if $A$ lies in $kC$, then by definition there exists a $J \in C$ such that $A - J \in (k-1)C$, hence if $A$ has the

maximal possible rank $kr$, then its rank must decrease by $r$ upon subtracting $J$: $\mathrm{rk}(A-J)=\mathrm{rk}(A)-r=(k-1)r$. It seems therefore reasonable, given an element $A \in \mathfrak{g}$ that we want to write as a sum of points on $C$, to look for $J \in C$ such that $\mathrm{rk}(A)-\mathrm{rk}(J)<\mathrm{rk}(A)$. The easy lemma below turns out to be an effective tool in the quest for such $J$.

Here, and in the rest of this paper, $V$ stands for the standard module of the classical Lie algebra under consideration, $V^*$ denotes the linear dual of $V$, and $\langle .,. \rangle$ is the natural pairing $V \times V^* \to K$. We identify $\mathfrak{gl}(V)$ with $V \otimes V^*$, and recall that under this identification the rank one elements of $\mathfrak{gl}(V)$ correspond to the tensors $y \otimes \eta$ with non-zero $y \in V$ and $\eta \in V^*$. Furthermore, for $A \in \mathfrak{gl}(V)$ we define the dual map $A^* \in \mathfrak{gl}(V^*)$ by $\langle x, A^*\xi \rangle = \langle Ax, \xi \rangle$.

**Lemma 2.1.** *For $A \in \mathfrak{gl}(V)$ and non-zero elements $y \in V, \eta \in V^*$ we have*

(1) $\mathrm{rk}(A - y \otimes \eta) < \mathrm{rk}(A)$ *if and only if* $y \in \mathrm{im}\, A, \ker \eta \supseteq \ker A$, *and* $\langle x, \eta \rangle = 1$ *for some (and hence for any)* $x \in A^{-1}y$*; and*

(2) $y \otimes \eta$ *lies in* $\mathfrak{sl}(V)$ *if and only if* $\langle y, \eta \rangle = 0$.

**Remark 2.2.** The conditions in the first statement are self-dual in $y$ and $\eta$, and can be rephrased more compactly as $\langle A^{-1}y, \eta \rangle = \{1\}$ or as $\langle y, (A^*)^{-1}\eta \rangle = \{1\}$.

*Proof.* The second statement is obvious. As for the first statement: if $y$ and $\eta$ satisfy the stated conditions, then $\ker(A-J)=\ker A \oplus Kx$, so that $\mathrm{rk}(A-J)<\mathrm{rk}(A)$ as claimed. Conversely, suppose that $\mathrm{rk}(A-J)<\mathrm{rk}(A)$, and let $x'$ be an element of $\ker(A-J) \setminus \ker A$. Then $(A-J)x' = Ax' - \langle x', \eta \rangle y = 0$ while $Ax' \neq 0$. We conclude that $c := \langle x', \eta \rangle$ is non-zero, so that we may set $x := x'/c$. Now $\langle x, \eta \rangle = 1$ and $Ax = y$ and, by a dual argument, $\eta$ lies in $\mathrm{im}\, A^*$, which is equivalent to $\ker \eta \supseteq \ker A$. □

Now first consider the Lie algebra $\mathfrak{sl}_n$ with standard module $V = K^n$. The highest root vector with respect to the usual Cartan and Borel subalgebras of $\mathfrak{sl}_n$ is the matrix with zeroes everywhere and a 1 in the upper right corner, hence of rank one. It is easy to see that the rank one elements of $\mathfrak{sl}_n$ form one orbit under the adjoint group $\mathrm{PSL}_n$: the minimal orbit $C$. Now we are ready to prove Theorem 1.1 in the case where $\mathfrak{g} = \mathfrak{sl}_n$.

*Proof of Theorem 1.1 for $\mathfrak{g} = \mathfrak{sl}_n$.* Let $A \in \mathfrak{sl}_n$ be non-zero; we show that there exists a $J \in C$ for which $\mathrm{rk}(A-J)=\mathrm{rk}(A)-1$. Indeed, $A$ induces a linear map $\bar{A}$ on the non-zero space $V/\ker A$, and we have $\mathrm{tr}\, \bar{A} = 0$. As $K$ has characteristic 0, the map $\bar{A}$, having trace zero, cannot be a non-zero scalar, so that there exists an $x \in V$ for which $x + \ker A$ is not a scalar multiple of $\bar{A}(x + \ker A)$. This means that $x$ does not lie in $KAx + \ker A$, hence there exists a linear function $\eta \in V^*$ that vanishes on $KAx + \ker A$ but has the value 1 on $x$. Now $J := Ax \otimes \eta$ lies in $\mathfrak{sl}_n$ and has $\mathrm{rk}(A-J)=\mathrm{rk}(A)-1$ by Lemma 2.1.

By induction, this proves that every element of $\mathfrak{sl}_n$ of rank $k$ lies in $kC$, for $k = 1, \ldots, n$. The inclusions $kC \subseteq (k+1)C$ and $0 \in 2C$, following from the fact that $C$ is a cone, conclude the proof that for $k \geq 2$ every element of rank *at most* $k$ lies in $kC$. Conversely, it was observed at the beginning of this section that $kC$ does not contain elements of rank higher than $k$. □

We proceed to prove Theorem 1.1 for the symplectic Lie algebra $\mathfrak{sp}_{2n}$. Let $V = K^{2n}$ be the standard $\mathfrak{sp}_{2n}$-module, and denote by $(.,.)$ the non-degenerate

skew bilinear form on $V$ defining $\mathfrak{sp}_{2n}$. Define the linear maps $\phi : V \to V^*$ and, for $A \in \mathfrak{gl}(V)$, $A^T : V \to V$ by $\langle x, \phi y\rangle = (x, y)$ and $(x, A^T y) = (Ax, y)$ for $x, y \in V$ and $\eta \in V^*$. We first describe the minimal $\mathrm{Sp}_{2n}$-orbit $C$ in $\mathfrak{sp}_{2n}$ in a convenient way.

**Lemma 2.3.** $C = \{x \otimes \phi x \mid x \in V \setminus \{0\}\} = \{J \in \mathfrak{sp}_{2n} \mid \mathrm{rk}\, J = 1\}$.

*Proof.* First, the highest root vector in $\mathfrak{sp}_{2n}$ (with respect to some choice of Cartan and Borel subalgebras) is easily seen to have rank one. Now a rank one element $J = y \otimes \eta$ of $\mathfrak{gl}_{2n}$ lies in $\mathfrak{sp}_{2n}$ if and only if $(Jx, z) = -(x, Jz)$ or, filling in the expression for $J$, if

$$\langle x, \eta\rangle (y, z) = -(x, y)\langle z, \eta\rangle$$

for all $x, z \in V$. By skewness of the form, this is clearly the case if $\eta = \phi(y)$. Conversely, for $z \in V$ fixed such that $(y, z) \neq 0$, the equation above shows that $\eta = c\phi(y)$ for some $c \neq 0$; hence if $d$ is a square root of $c$, then $J = y \otimes \eta = dy \otimes \phi(dy)$. This shows that the second set of the lemma is equal to the third, and that they contain $C$. Finally, the transitivity of $\mathrm{Sp}_{2n}$ on $V \setminus \{0\}$ implies the transitivity of $\mathrm{Sp}_{2n}$ on the second set of the lemma, and this concludes the proof. □

*Proof of Theorem 1.1 for* $\mathfrak{sp}_{2n}$. Let $A \in \mathfrak{sp}_{2n}$ be non-zero. We show that there exists an $x \in V$ such that $A - (Ax \otimes \phi(Ax))$ has rank $\mathrm{rk}(A) - 1$. By Lemma 2.1 this is the case if and only if $\ker \phi(Ax) \supseteq \ker A$ and $\langle x, \phi(Ax)\rangle = 1$. The first condition holds for all $x \in V$, as $Az = 0$ implies

$$\langle z, \phi(Ax)\rangle = (z, Ax) = -(Az, x) = 0$$

by virtue of $A^T = -A$. Hence, we are left to show that there exists an $x \in V$ for which $\langle x, \phi(Ax)\rangle = (x, Ax) \neq 0$; rescaling $x$ will then make this scalar 1. Suppose, on the contrary, that $(x, Ax) = 0$ for all $x$. Then we have for all $x, y \in V$:

$$0 = (x + y, A(x + y)) = (x, Ax) + (y, Ay) + (x, Ay) + (y, Ax) = 2(x, Ay),$$

so that $A = 0$, which contradicts our assumption that $\mathrm{rk}(A)$ be greater than 1.

By induction, this shows that for $k = 1, \ldots, 2n$ every element of $\mathfrak{sp}_{2n}$ of rank $k$ lies in $kC$. As in the case of $\mathfrak{sl}_n$, the inclusions $kC \subseteq (k+1)C$ and $0 \in 2C$, together with the fact that $kC$ cannot contain elements of rank higher than $k$, conclude the proof. □

## 3. The main course: $\mathfrak{o}_n$

Now we come to the more intricate part of this paper: the secant varieties of the minimal orbit $C$ of $\mathrm{SO}_n$ on its Lie algebra $\mathfrak{o}_n$. Unlike in the cases of $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$, the sums $kC$ are in general not closed, and only their closures are described explicitly here. The approach, though, is the same as for $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$: we try to decrease the rank of a given element of $\mathfrak{o}_n$ by subtracting an appropriate element of $C$. How this rank reduction works for $\mathfrak{o}_n$, and why it comes short of characterising the sets $kC$ completely, is explained in Subsection 3.1. Subsections 3.2, 3.3, and 3.4 are devoted to determining $\overline{2C}$, $\overline{3C}$, and $\overline{kC}$ for $k \geq 4$, respectively. In Subsection 3.2 we find that the complement of $2C$ in $\overline{2C}$ is a single nilpotent orbit, which discovery motivates the discussion of nilpotent orbits in Subsection 3.5.

3.1. **The minimal orbit and rank reduction.** We retain the notation $\phi$ and $A^T$ from Section 2; only now they are defined with respect to the non-degenerate *symmetric* bilinear on $V = K^n$ defining the Lie algebra $\mathfrak{o}_n$. Recall that, for any $A \in \mathfrak{o}_n$ and $\lambda \in K$, the numbers $\lambda$ and $-\lambda$ have the same (geometric and algebraic) multiplicity among the eigenvalues of $A$; moreover, $\mathrm{rk}(A)$ is even. The following lemma implies that every $A \in \mathfrak{o}_n$ of rank $2k$ is the sum of $k$ rank two elements of $\mathfrak{o}_n$.

**Lemma 3.1.** *Let $A \in \mathfrak{o}_n$, and let $J \in \mathfrak{gl}_n$ of rank one be such that $\mathrm{rk}(A - J) = \mathrm{rk}(A) - 1$. Then $\mathrm{rk}(A - (J - J^T)) = \mathrm{rk}(A) - 2$.*

The proof of this lemma uses the useful identities $(y \otimes \eta)^T = \phi^{-1}\eta \otimes \phi y$ $(y \in V, \eta \in V^*)$ and $\phi A^T = A^*\phi$ $(A \in \mathfrak{gl}_n)$, whose proofs are straightforward.

*Proof.* By Lemma 2.1, there exist $x \in V$ and $\xi \in V^*$ such that $J = Ax \otimes A^*\xi$ and $\langle Ax, \xi\rangle = 1$; note that then $\ker(A - J) = \ker A \oplus Kx$. We have

$$J^T = \phi^{-1}A^*\xi \otimes \phi Ax = A^T\phi^{-1}\xi \otimes (A^T)^*\phi x = A\phi^{-1}\xi \otimes A^*\phi x,$$

where the third step is justified by $A^T = -A$. In particular, we find that $\ker J^T$, too, contains $\ker A$, so that $\ker A \subseteq \ker(A - J + J^T)$. Moreover, we have

$$(A - J + J^T)x = J^T x = \langle x, A^*\phi x\rangle A\phi^{-1}\xi = (Ax, x)A\phi^{-1}\xi = 0,$$

while $Ax \neq 0$. (In the last step we used $(Ax, x) = (x, A^T x) = -(x, Ax) = -(Ax, x)$.) Hence, $\mathrm{rk}(A - J + J^T)$ is strictly smaller than $\mathrm{rk}(A)$; but as $A - J + J^T$ is skew symmetric, its ranks is even, hence equal to $\mathrm{rk}(A) - 2$. □

If $C$ would contain all elements of $\mathfrak{o}_n$ of rank 2, then we would have $mC = \mathfrak{o}_n$ by this lemma. However, $C$ is smaller; to characterise it we first describe the rank-two-elements of $\mathfrak{o}_n$.

**Proposition 3.2.** *For any 2-dimensional subspace $W = \langle y_1, y_2\rangle_K$ of $V$, the space $\{A \in \mathfrak{o}_n \mid \mathrm{im}\, A \subseteq W\}$ is one-dimensional and spanned by $y_1 \otimes \phi(y_2) - y_2 \otimes \phi(y_1)$.*

The proof of this proposition uses another easy observation; namely, that for any $A \in \mathfrak{o}_n$ the kernel of $A$ is the orthogonal complement of $\mathrm{im}\, A$ with respect to $(.,.)$; we denote this orthogonal complement by $(\mathrm{im}\, A)^\perp$.

*Proof.* Let $A \in \mathfrak{o}_n \setminus \{0\}$ have image contained in, and hence equal to, $W$; and let $x_1 \in V$ be such that $Ax_1 = y_1$. Then we have $(x_1, y_1) = (x_1, Ax_1) = 0$ by the skewness of $A$, so that $(x_1, y_2) = 0$ would imply $x_1 \in (\mathrm{im}\, A)^\perp = \ker A$, a contradiction, hence we may set $\alpha := 1/(x_1, y_2)$. Furthermore, $y_2^\perp \supseteq \ker A$, so that $J := \alpha y_1 \otimes \phi(y_2) \in \mathfrak{gl}_n$ satisfies the condition of Lemma 3.1. Then that lemma implies $A = \alpha(y_1 \otimes \phi(y_2) - y_2 \otimes \phi(y_1))$, as claimed. □

Proposition 3.2 has the following interesting consequence.

**Corollary 3.3.** *For each $k \in \{0, 1, 2\}$, the group $\mathrm{SO}_n$ acts transitively on the set $O_k := \mathbb{P}\{A \in \mathfrak{o}_n \mid \mathrm{rk}(A) = 2$ and $(.,.)|_{\mathrm{im}\, A}$ has rank $k\} \subseteq \mathbb{P}\mathfrak{g}$.*

*Proof.* It is not hard to see that $\mathrm{SO}_n$ acts transitively on the 2-dimensional subspaces of $V$ on which $(.,.)$ has rank $k$, and now the proposition can be applied. □

The following corollary identifies $\mathbb{P}C$ with $O_0$.

**Corollary 3.4.** *The set $C$ consists of all $A \in \mathfrak{o}_2$ with $\mathrm{rk}(A) \leq 2$ and $\mathrm{im}\, A$ isotropic with respect to $(.,.)$. The latter condition is equivalent, for $A \in \mathfrak{o}_2$, to $A^2 = 0$.*

*Proof.* By Corollary 3.3, it suffices to check that the highest root vector of $\mathfrak{o}_n$ with respect to some choice of Borel and Cartan subalgebras has the stated properties, which is straightforward. As for the second statement: the radical of $(.,.)|_{\mathrm{im}\, A}$ on $\mathrm{im}\, A$ is exactly $\ker A \cap \mathrm{im}\, A$, hence all of $\mathrm{im}\, A$ if and only if $A^2 = 0$. □

We are now ready to state and prove our main rank reduction argument in the orthogonal case.

**Proposition 3.5.** *Let $A \in \mathfrak{o}_n$ be of rank $\geq 4$. Then there exists a $J \in C$ such that $\mathrm{rk}(A - J) = \mathrm{rk}(A) - 2$.*

*Proof.* On $\mathrm{im}\, A$ we have two bilinear forms: the restriction of $(.,.)$, and a second form $(.|.)$ defined by $(Ax_1|Ax_2) = (x_1, Ax_2)$; we continue to use $\perp$ only for 'perpendicular with respect to $(.,.)$'. The second form is well-defined as $\ker A \perp \mathrm{im}\, A$ and skew-symmetric because

$$(Ax_2|Ax_1) = (x_2, Ax_1) = (A^T x_2, x_1) = -(Ax_2, x_1) = -(x_1, Ax_2) = -(Ax_1|Ax_2).$$

Moreover, $(.|.)$ is non-degenerate, as $(Ax_1|Ax_2) = 0$ for all $x_1$ implies $x_1 \perp Ax_2$ for all $x_1$, i.e., $Ax_2 = 0$. We may now apply Lemma 3.6 below to find a 2-dimensional subspace $U$ of $\mathrm{im}\, A$ that is isotropic with respect to $(.,.)$ but not with respect to $(.|.)$. Choose a basis $y_1, y_2$ of $U$ such that $(y_1|y_2) = 1$, and set $J := y_1 \otimes \phi(y_2) - y_2 \otimes \phi(y_1)$. Then $\mathrm{im}\, J$ is two-dimensional and isotropic with respect to $(.,.)$, so $J$ lies in $C$ by Corollary 3.4. Furthermore, $\ker J = U^\perp$ contains $\ker A = (\mathrm{im}\, A)^\perp$, and if $x_1 \in A^{-1} y_1$, then

$$(A - J)x_1 = y_1 - (x_1, y_2)y_1 + (x_1, y_1)y_2 = y_1 - (y_1|y_2)y_1 + (y_1|y_1)y_2 = 0,$$

so that $\mathrm{rk}(A - J)$ is strictly smaller than $\mathrm{rk}(A)$; we conclude that $J$ has the required properties. □

The proof above uses the following observation on bilinear forms.

**Lemma 3.6.** *Let $W$ be a $K$-vector space of finite dimension $\geq 4$ equipped with a (possibly degenerate) symmetric bilinear form $B_1$ and a non-degenerate skew-symmetric bilinear form $B_2$. Then there exists a 2-dimensional subspace of $W$ that is isotropic with respect to $B_1$ but not with respect to $B_2$.*

The following proof, which is considerably shorter than our original proof, was suggested by Jochen Kuttler.

*Proof.* Suppose, on the contrary, that all 2-dimensional $B_1$-isotropic subspaces of $W$ are $B_2$-isotropic, and note that then all $B_1$-isotropic subspaces of *any* dimension are $B_2$-isotropic. We may choose a basis $e_1, \ldots, e_d$ of $W$ such that $B_1(x, y) = \sum_{j=1}^{l} x_j y_j$, where $l \leq d$ is the rank of $B_1$. If $l = 0, 1$, or $2$, then the subspace of codimension 1 defined by the equation $x_1 = 0, x_1 = 0$, or $x_2 = ix_1$, respectively, is isotropic with respect to $B_1$, and hence with respect to $B_2$. On the other hand, any $B_2$-isotropic subspace of $W$ has dimension at most $\dim(W)/2$, so that $\dim(W) - 1 \leq \dim(W)/2$, a contradiction to $\dim(W) \geq 4$.

Hence $l$ is at least 3. Now consider the quadric $Q_1 := \{x \in W \mid B_1(x, x) = 0\}$. It is easy to find linearly independent vectors $w_1, w_2, w_3$ on $Q_1$ such that $B_1(w_j, w_k) \neq 0$ for all distinct $j, k \in \{1, 2, 3\}$—for example, $w_1 = e_1 + ie_2, w_2 = e_1 + ie_3$, and

$w_3 = e_2 + ie_3$. Moreover, $Q_1$ spans $W$ and we may find $w_4, \ldots, w_d \in Q_1$ such that $w_1, w_2, \ldots, w_d$ is a basis of $W$; we write $w_j^\perp$ for $\{w \in W \mid B_1(w_j, w) = 0\}$. For each $j$ and any $w \in Q_1 \cap w_j^\perp$ the space $Kw_j + Kw$ is $B_1$-isotropic, so that $B_2(w_j, w) = 0$ by assumption. As, moreover, the restriction of $B_1$ to $w_j^\perp$ has rank at least two, we find that $Q_1 \cap w_j^\perp$ spans $w_j^\perp$, so that $B_2(w_j, w) = 0$ for any $w \in w_j^\perp$. In other words, the linear function $B_2(w_j, .)$ is equal to $c_j B_1(w_j, .)$ for some $c_j \in K$, so that if $A_1, A_2$ are the matrices of $B_1, B_2$ with respect to $w_1, \ldots, w_d$, then

$$A_2 = \operatorname{diag}(c_1, \ldots, c_d) A_1.$$

As $A_2$ is skew (with respect to transposition in the main diagonal) and $A_1$ is symmetric, we find that $c_j a_{jk} = -c_k a_{jk}$ for all $j, k = 1, \ldots, d$. By construction $a_{jk} \neq 0$ for distinct $j, k \in \{1, 2, 3\}$, and we find that $c_j = -c_k$ for all such $j, k$. This readily implies that $c_1 = c_2 = c_3 = 0$, so that $A_2$ is singular; but this contradicts the non-degeneracy of $B_2$. $\square$

We can now prove Theorem 1.7; from the proof it will become clear why the rank reduction of Proposition 3.5 does not suffice to characterise the secant varieties of $C$ completely.

*Proof of Theorem 1.7.* By Proposition 3.5 and induction, it suffices to prove that every element of $\mathfrak{o}_n$ having rank 2 lies in $4C$. By Corollary 3.3 (and the fact that $4C$ is, of course, a cone) it suffices to prove this for particular representatives of the projective orbits $O_k$ ($k = 0, 1, 2$) mentioned in that corollary. For $k = 0$ we have $O_0 = \mathbb{P}C$, so there is nothing to prove. For $k = 1, 2$ let $y_1, y_2, y_3, y_4$ be linearly independent isotropic vectors in $V$ satisfying $(y_1, y_3) = (y_2, y_4) = 1$ and $\langle y_1, y_3 \rangle_K \perp \langle y_2, y_4 \rangle_K$ (such vectors exist). Then a representative of $O_1$ is

$$(y_1 + y_3) \otimes \phi(y_2) - y_2 \otimes \phi(y_1 + y_3),$$

which can be written as

$$(y_1 \otimes \phi(y_2) - y_2 \otimes \phi(y_1)) + (y_3 \otimes \phi(y_2) - y_2 \otimes \phi(y_3)) \in 2C.$$

Similarly, a representative of $O_2$ is

$$(y_1 + y_3) \otimes \phi(y_2 + y_4) - (y_2 + y_4) \otimes \phi(y_1 + y_3),$$

which equals

$$\begin{gathered}(y_1 \otimes \phi(y_2) - y_2 \otimes \phi(y_1)) + (y_1 \otimes \phi(y_4) - y_4 \otimes \phi(y_1)) \\ +(y_3 \otimes \phi(y_2) - y_2 \otimes \phi(y_3)) + (y_3 \otimes \phi(y_4) - y_4 \otimes \phi(y_3)) \in 4C.\end{gathered}$$

$\square$

One may think, now, that a representative of $O_2$ could already lie in $kC$ for $k = 2$ or 3—but this is not the case. Indeed, as we shall see in Subsection 3.3, such a representative does not even lie in $\overline{3C}$. This serves to show that the secant varieties of the minimal orbit in $\mathfrak{o}_n$ are considerably more complicated than those of the minimal orbits in $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$.

3.2. **The first secant variety.** The first secant variety $\mathbb{P}(\overline{2C} \setminus \{0\})$ of the minimal orbit in *any* simple Lie algebra is described in [14] as the union of a single (projective) semisimple orbit and several nilpotent orbits. We reprove this statement here for $\mathfrak{o}_n$; first, because our method is different from that of Kaji *et al* and also applies to the second secant variety, and second, because we want to determine the complement $\overline{2C} \setminus 2C$ explicitly.

Before stating our characterisation of $\overline{2C}$, we recall that the closed subvariety

$$R_k := \{A \in \mathfrak{o}_n \mid \operatorname{rk} A \leq 2k\}$$

is irreducible for all $k$—this follows, for instance, from [11, Lemma 4.2.4(3)]—and we recall from the introduction the notation $S_2$ for the set of rank 4 semisimple elements having non-zero eigenvalues $a, a, -a, -a$.

**Proposition 3.7.** *The affine variety*

$$M := \{A \in \mathfrak{o}_n \mid \operatorname{rk}(A) \leq 4 \text{ and } A^3 = \lambda A \text{ for some } \lambda \in K\}$$

*has two irreducible components, namely $R_1$ and $\overline{2C}$. Furthermore, $\overline{2C}$ is equal to $\overline{S_2}$.*

Recall, for the proof of this proposition, the notation $\mathcal{O}[\mathbf{d}]$ for the nilpotent $\mathrm{O}_n$-orbit on $\mathfrak{o}_n$ corresponding to the partition $\mathbf{d}$ of $n$, where the even entries of $\mathbf{d}$ are supposed to have even multiplicities. We work with $\mathrm{O}_n$ here, rather than with the adjoint group $\mathrm{SO}_n$, not to have to distinguish between the two $\mathrm{SO}_n$-orbits corresponding to *very even* partitions [7, 16]. Indeed, as both groups have the same minimal orbit $C = \mathcal{O}[2, 2, 1^{n-4}]$, this subtlety is immaterial to us.

We will not be able to avoid, in what follows, some explicit matrix computations. In these computations we always take for $(.,.)$ the symmetric form given by $(x, y) = \sum_{i=1}^{n} x_i y_{n+1-i}$ with respect to the standard basis of $V = K^n$. The elements of $\mathfrak{o}_n$ are then skew symmetric about the *skew* diagonal running from position $(1, n)$ to position $(n, 1)$.

*Proof of Proposition 3.7.* For $J_1, J_2 \in C$ we have

$$(J_1 + J_2)^3 = J_1 J_2 J_1 + J_2 J_1 J_2,$$

where we use that $J_i^2 = 0$ for $i = 1, 2$ (Corollary 3.4). The map $J_1 J_2 J_1$ is skew-symmetric and its image is contained in $\operatorname{im} J_1$, hence by Proposition 3.2 $J_1 J_2 J_1 = c_1 J_1$ for some $c_1 \in K$. Similarly, $J_2 J_1 J_2 = c_2 J_2$ for some $c_2 \in K$. If $J_1 J_2 = 0$, then $c_1 = c_2 = 0$ and $J_1 + J_2 \in M$ (with $\lambda = 0$). Otherwise, let $x \in V$ be such that $J_1 J_2 x \neq 0$. Then

$$c_2 J_1 J_2 x = J_1 (J_2 J_1 J_2) x = (J_1 J_2 J_1) J_2 x = c_1 J_1 J_2 x,$$

so that $c_1 = c_2$. This shows that $J_1 + J_2$ lies in $M$ (with $\lambda = c_1$), so that $\overline{2C} \subseteq M$. The inclusion $R_1 \subseteq M$ is immediate: an element $A$ of $R_1$ is either semisimple with non-zero eigenvalues $a, -a$, so that $A \in M$ (with $\lambda = a^2$), or it is nilpotent of nilpotence degree at most 3, and then $A$ also lies in $M$ (with $\lambda = 0$).

Conversely, let $A$ be in $M$ and let $\lambda \in K$ be such that $A^3 = \lambda A$. If $\lambda = 0$, then $A^3 = 0$, which together with the condition that $\operatorname{rk}(A)$ be at most 4 shows that $A$ lies in a nilpotent orbit corresponding to one of the partitions $[3, 3, 1^{n-6}]$, $[3, 2, 2, 1^{n-7}]$, $[3, 1^{n-3}]$, $[2, 2, 2, 2, 1^{n-8}]$, $[2, 2, 1^{n-4}]$, or $[1^n]$. The first among these is greater than all of the other five in the usual order on partitions [7], so that the corresponding orbit closure contains the other five nilpotent orbits.

Suppose, on the other hand, that $\lambda \neq 0$, and let $a$ be a square root of $\lambda$. Then $A$ is a zero of the square-free polynomial $t(t-a)(t+a)$, hence semisimple. There are three possibilities: either $A = 0$, or $A \in R_1$ with non-zero eigenvalues $\pm a$, or $A \in S_2$ with eigenvalues $a, a, -a, -a$. Together with the above discussion of the nilpotent orbits in $M$ this implies $M = R_1 \cup S_2 \cup \overline{\mathcal{O}}[3,3,1^{n-6}]$. We shall show that the last two terms are contained in $\overline{2C}$, so that

$$M = R_1 \cup \overline{2C};$$

as $R_1$ and $\overline{2C}$ are both irreducible and neither of these sets is contained in the other, this implies the first statement of the proposition. Moreover, the above shows that the complement of $S_2$ in $\overline{2C}$ equals $(R_1 \cap \overline{2C}) \cup \overline{\mathcal{O}}[3,3,1^{n-6}]$, so that $S_2$ is open, and hence dense, in $\overline{2C}$—which proves the second statement of the proposition.

Suppose, therefore, that $A$ lies in $\mathcal{O}[3,3,1^{n-6}]$. Then $A$ is conjugate to an $n \times n$-matrix that has a $6 \times 6$ block

$$\begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & & & \\ & & & 0 & -1 & \\ & & & & 0 & -1 \\ & & & & & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & -1 \\ & & & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & & & & \\ & 0 & 1 & & & \\ & & 0 & & & \\ & & & 0 & -1 & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}$$

in the middle, and zeroes elsewhere (off-diagonal zeroes are omitted). Now both matrices on the right-hand side of the equality lie in $C$: they have rank 2 and isotropic images. Therefore, $A$ lies in $2C$ and $\overline{\mathcal{O}}[3,3,1^{n-6}] \subseteq 2C$.

Next assume that $A$ has rank 4 and is semisimple with non-zero eigenvalues $a, a, -a, -a$. Then $A$ is conjugate to a matrix with a $4 \times 4$-block

$$a \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} = \frac{a}{2} \begin{bmatrix} 1 & & 1 & \\ & 1 & & -1 \\ -1 & & -1 & \\ & 1 & & -1 \end{bmatrix} + \frac{a}{2} \begin{bmatrix} 1 & & -1 & \\ & 1 & & 1 \\ 1 & & -1 & \\ & -1 & & -1 \end{bmatrix}$$

in the middle and zeroes elsewhere. One readily verifies that the two terms on the left-hand side lie in $C$, so that $A \in 2C$. As explained above, this concludes the proof of the proposition. $\square$

Now that we have identified $\overline{2C}$ as the irreducible component $\overline{S_2}$ of $M$—and hence proved the first part of Theorem 1.3—we investigate the set $2C$ itself. It is easy to verify, like we did in the proof above for $\mathcal{O}[3,3,1^{n-6}]$, that the nilpotent orbits $\mathcal{O}[3,1^{n-3}]$, $\mathcal{O}[2,2,2,2,1^{n-8}]$, $\mathcal{O}[2,2,1^{n-4}]$, and $\mathcal{O}[1^n]$ in $\overline{S_2}$ all lie in $2C$ (in fact, this follows from the computations in Subsection 3.5 below), as does $S_2$ by the explicit computation in the proof above. Thus we find that $\overline{2C} \setminus 2C$ is contained in $\mathcal{O}[3,2,2,1^{n-7}]$. Note that an element $A$ from this nilpotent orbit has $\mathrm{rk}(A) = 4$ and $\mathrm{rk}(A^2) = 1$. The following lemma shows that an element of $2C$ cannot have this property, thus proving Proposition 1.5.

**Lemma 3.8.** *If $A \in 2C$ has rank 4, then $\mathrm{rk}(A^2)$ is even.*

*Proof.* By Corollary 3.4 we may write

$$A = y_1 \otimes \phi(y_2) - y_2 \otimes \phi(y_1) + y_3 \otimes \phi(y_4) - y_4 \otimes \phi(y_4),$$

where $\langle y_1, y_2 \rangle_K$ and $\langle y_3, y_4 \rangle_K$ are isotropic. By the condition that $\mathrm{rk}(A)$ be 4, the vectors $y_1, y_2, y_3, y_4$ are a basis of $\mathrm{im}\, A$. The matrix of $(.,.)|_{\mathrm{im}\, A}$ with respect to this

basis is of the form

$$\begin{bmatrix} 0 & M \\ M^t & 0 \end{bmatrix},$$

where $M^t$ is the (ordinary) transpose of the $2 \times 2$-matrix $M$. We conclude that $\operatorname{rk} A^2 = \operatorname{rk}(.,.)|_{\operatorname{im} A} = 2 \operatorname{rk} M$. $\square$

3.3. **The second secant variety.** To characterise $\overline{3C}$ we proceed as in the first part of the proof of Proposition 3.7: we take three arbitrary elements $J_1, J_2, J_3$ of $C$, and use the relations provided by Proposition 3.2 to find a polynomial annihilating $J_1 + J_2 + J_3$. Conversely, we show that semisimple elements having a characteristic polynomial of that form do indeed lie in $3C$.

**Proposition 3.9.** *Any element of $\overline{3C}$ is annihilated by a polynomial of the form*

$$t(t-a)(t-b)(t-c)(t+a)(t+b)(t+c)$$

*for some $a, b, c \in K$ with $a + b + c = 0$.*

*Proof.* The set of matrices in $\mathfrak{o}_n$ that are annihilated by such a polynomial, is closed, so that it suffices to prove the proposition for elements of $3C$. Let therefore $J_1, J_2, J_3$ be elements of $C$. From the proof of Proposition 3.7 we know that there exist constants $c_{ik} = c_{ki}$ such that

$$J_i J_k J_i = c_{ik} J_i \text{ for all } i, k \in \{1, 2, 3\},\ i \neq k.$$

Similarly, there exists a $c \in K$ with

$$J_1(J_2J_3J_1J_2J_3 + J_3J_2J_1J_3J_2)J_1 = cJ_1;$$

indeed, the matrix between brackets is an element of $\mathfrak{o}_n$, so that the matrix on the left-hand side lies in $\mathfrak{o}_n$. Furthermore, its image is contained in $\operatorname{im} J_1$, whence the existence of such a $c$ follows from Proposition 3.2. In fact, one can show that the same $c$ satisfies the above relation with $1, 2, 3$ permuted cyclically. Using these relations, a straightforward calculation shows that

$$tp(t) \text{ with } p(t) := (t^3 - (c_{12} + c_{13} + c_{23})t)^2 - c - 2c_{12}c_{13}c_{23}$$

annihilates $J_1 + J_2 + J_3$. Now we need only check that $p$ has the desired form. To this end, let $\mu$ be a square root of $c + 2c_{12}c_{13}c_{23}$, so that $p$ factorises into

$$p(t) = (t^3 - (c_{12} + c_{13} + c_{23})t + \mu)(t^3 - (c_{12} + c_{13} + c_{23})t - \mu).$$

The first of these factors lacks a term with monomial $t^2$; hence, the sum of its zeroes $a, b, c$ is 0. The second factor has zeroes $-a, -b, -c$, and this concludes the proof of the proposition. $\square$

**Remark 3.10.** The polynomial $tp(t)$ appearing in the proof above was found as follows: consider the free associative algebra $F$ (with one) over the ground field $K(c_{12}, c_{13}, c_{23}, c)$ with generators $J_1, J_2, J_3$ and let $I$ be the ideal generated by the relations appearing in the proof above. Then a (non-commutative) Gröbner basis computation of $I$ shows that $F/I$ has dimension 37, and the polynomial $tp(t)$ is the minimal polynomial of $J_1 + J_2 + J_3$ in this quotient. For this computation we used the `GAP`-package `GBNP` written by Cohen and Gijsbers [5, 9] (with concrete values for the $c_{ij}$ and $c$), together with some *ad hoc* programming of our own in `Mathematica`.

A partial converse to the proposition above is the following lemma, in whose proof we compute with respect to the fixed bilinear form of Subsection 3.2.

**Lemma 3.11.** *For all $a, b \in K$, any semisimple element of $so_n$ whose eigenvalues (with multiplicities) are 0 ($n-6$ times) and $a, b, -a-b, -a, -b, a+b$ for some $a, b \in K$, lies in $3C$.*

*Proof.* Let $A$ be such an element; we may suppose that $A$ is non-zero. Then the numbers $a, b, -a-b$ are not all equal, and by permuting them we may assume that $a \neq b$. Now $A$ is conjugate to an $n \times n$-matrix having zeroes everywhere except for a $6 \times 6$-block in the middle, which is of the form

$$\begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & a+b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a-b & 0 & 0 \\ 0 & 0 & 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 & 0 & -a \end{bmatrix} = \frac{1}{b-a} \begin{bmatrix} ab & ab & 0 & 0 & 0 & 0 \\ -ab & -ab & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ab & -ab \\ 0 & 0 & 0 & 0 & ab & -ab \end{bmatrix}$$
$$+ \frac{1}{b-a} \begin{bmatrix} -a^2 & -ab & 0 & 0 & 0 & 0 \\ ab & b^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & (b+a)(b-a) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(b+a)(b-a) & 0 & 0 \\ 0 & 0 & 0 & 0 & -b^2 & ab \\ 0 & 0 & 0 & 0 & -ab & a^2 \end{bmatrix}.$$

The first term on the right-hand side lies in $C$, as it has rank 2 and its image is isotropic. We claim that the second term on the right-hand side, which we denote by $B$, lies in $2C$. To see this, note that $(b, -a, 0, 0, 0, 0)^t$, $(0, 0, 1, 0, 0, 0)^t$, and $(a, -b, 0, 0, 0, 0)^t$ are eigenvectors of $B$ with eigenvalues $0, a+b$, and $a+b$, respectively. It follows that $B$ has rank 4 and that $-a-b$, as well, has multiplicity two among the eigenvalues of $B$; hence $B$ lies in $2C$ by Propositions 1.5 and 3.7, and $A$ lies in $3C$. □

To finish our characterisation of $\overline{3C}$, recall that $R_k$ is the set of all elements of $\mathfrak{o}_n$ having rank $\leq 2k$. We now need an argument why $S_3$, the set of all semisimple elements in $R_3$ having 6 *distinct* eigenvalues $a, b, c, -a, -b, -c$ such that $a+b+c = 0$, is dense in $\overline{3C}$. The following lemma will provide such an argument.

**Lemma 3.12.** *For any $k$, the subset $T_k$ of $R_k$ consisting of all elements having $2k$ distinct non-zero eigenvalues, is open in $R_k$.*

*Proof.* An element of $R_k$ lies in $R_k \setminus T_k$ if and only if it has a characteristic polynomial of the form

$$t^{n-2k}(t^2 - a_1)^2(t^2 - a_2)(t^2 - a_3)\dots(t^2 - a_{k-1})$$

for some $a_1, \dots, a_{k-1} \in K$. The map $K^{k-1} \to K^{n+1}$ sending $(a_1, \dots, a_{k-1})$ to the coefficients of the monomials $t^i$ in the polynomial above has a closed image $Y$, and $R_k \setminus T_k$ is the inverse image of $Y$ under the polynomial map sending a matrix to the coefficients of its characteristic polynomial. □

From Proposition 3.9 and Lemma 3.11 we have

$$S_3 = T_3 \cap \overline{3C}.$$

By the lemma above, this set is open, and hence dense, in $\overline{3C}$, so that $\overline{3C} = \overline{S_3}$ as claimed in Theorem 1.3.

3.4. **Higher secant varieties.** After reading the discussion of $\overline{2C}$ and $\overline{3C}$, one could think that to describe the sets $\overline{kC}$ for $k \geq 4$, we must consider the quotient of the free algebra generated by $J_1, \ldots, J_k$ by the ideal generated by all relations that can be inferred from Proposition 3.2, i.e., those of the form $J_i J_k J_i = c_{ik} J_i$ appearing in the proof of Proposition 3.7, those reflecting that $J_i(J_k J_l J_i J_k J_l + J_l J_k J_i J_l J_k J_i) J_i$ is a scalar multiple of $J_i$ (where the scalar does not change if we permute $i, k, l$ cyclically; this relation appears in the proof of Proposition 3.9), and similar relations, such as: $J_i(J_k J_l J_m + J_m J_l J_k) J_i$ is a scalar multiple of $J_i$. While this quotient algebra may be interesting in itself—is it always finite-dimensional? Gröbner Basis computations seem to end in an endless loop already for $k = 4$—it does, surprisingly enough, not play an important role in determining the higher secant varieties of $C$. The following proposition explains why.

**Proposition 3.13.** *The set $4C$ contains a dense subset of $R_4$.*

Together with the obvious inclusion $4C \subseteq R_4$, this proposition implies $\overline{4C} = R_4$. Using Proposition 3.5 we then find $\overline{kC} = R_{2k}$ for all $k \geq 4$, as claimed in Theorem 1.3.

*Proof.* Let $a_1, a_2, a_3, a_4$ be variables. It suffices to prove that the diagonal matrix

$$A = \mathrm{diag}\{a_1, \ldots, a_4, -a_4, \ldots, -a_1\}$$

lies in $4C(K(a_1, \ldots, a_4))$, i.e., 4 times the minimal orbit in $\mathfrak{o}_8$ with coordinates in $K(a_1, \ldots, a_4)$. Indeed, if this is the case, then a generic semisimple element of $R_4$ lies in $4C$, and these elements are dense in $R_4$ by Lemma 3.12. Define the expressions

$$\begin{aligned}
r_1 &:= 0, & s_1 &:= 1,\\
r_2 &:= a_3(a_1^2 - (a_2 + a_3 + a_4)^2), & s_2 &:= 4(a_2 + a_3)(a_3 + a_4),\\
r_3 &:= -a_4(a_1^2 - (-a_2 + a_3 + a_4)^2), & s_3 &:= 4(a_3 + a_4)(-a_2 + a_4),\\
r_4 &:= -a_2(a_1^2 - (-a_2 - a_3 + a_4)^2), \text{ and} & s_4 &:= 4(-a_2 + a_4)(-a_2 - a_3);
\end{aligned}$$

and note that the transformation $a_2 \mapsto a_3 \mapsto a_4 \mapsto -a_2$ cyclically permutes $s_2, s_3, s_4$, and does the same with $r_2, r_3, r_4$ up to a change of sign. Now set

$$\begin{aligned}
y_1 &:= (0, r_4, 0, r_3, 0, r_2, 0, r_1)^t, \qquad y_2 := (\frac{1}{s_1}, 0, \frac{1}{s_2}, 0, \frac{1}{s_3}, 0, \frac{1}{s_4}, 0)^t, \text{ and}\\
J &:= y_1(y_2^t F) - y_2(y_1^t F),
\end{aligned}$$

where $F = (\delta_{i+j,9})_{ij}$ is the $8 \times 8$-matrix representing the form $(.,.)$. By construction $J$ lies in $\mathfrak{o}_n(K(a_1, \ldots, a_4))$ and has rank 2. It is easy to see that $y_1^t F y_1 = y_2^t F y_2 = 0$, and a straightforward computation shows that $y_1^t F y_2 = \sum_{i=1}^4 \frac{r_i}{s_i}$ is zero, as well. This shows that $\mathrm{im}\, J$ is isotropic, hence $J$ lies in $C(K(a_1, \ldots, a_4))$. A direct computation (preferably by a computer algebra system; we used `Mathematica`) shows that $A - J$ is semisimple with eigenvalues

$$0, 0, \mp a_1, \pm\frac{1}{2}(a_1 - a_2 + a_3 - a_4), \pm\frac{1}{2}(a_1 + a_2 - a_3 + a_4);$$

if we take the upper one of the two signs in each of the last three eigenvalues, then they add up to zero, so that $A - J \in 3C(K(a_1, \ldots, a_4))$ by Lemma 3.11. We conclude that $A$ lies in $4C(K(a_1, \ldots, a_4))$ as claimed. $\square$

**Remark 3.14.** By studying the computations needed for the proof above, it should be straightforward to prove that *any* semisimple element of $R_4$ lies in $4C$. Furthermore, the computation proving Lemma 3.11 is easily modified to a proof that for any semisimple $A \in \mathfrak{o}_n$ of rank $2k,\ k \geq 2$, there exists an element $J \in C$ such that $\mathrm{rk}(A - J) = \mathrm{rk}(A) - 2$ and $A - J$ is again semisimple (this is a 'semisimple version' of Proposition 3.5). Summarising, this would prove that for $k \geq 4$ any semisimple element of $R_k$ lies in $kC$.

3.5. **Nilpotent orbits.** As we have seen in Subsection 3.2, the set $\overline{2C} \setminus 2C$ consists of the single nilpotent orbit $\mathcal{O}[3, 2, 2, 1^{n-7}]$. This motivates the question of what the minimal $k$ is such that a given nilpotent orbit $\mathcal{O}$ lies in $kC$. We will see that usually, this $k$ is just half the rank of $\mathcal{O}$. However, this is not true for the orbits $\mathcal{O}[3, 1^{n-3}]$, $\mathcal{O}[3, 2, 2, 1^{n-3}]$, and $\mathcal{O}[5, 1^{n-5}]$. Furthermore, it remains an open question what the minimal $k$ is for partitions whose odd entries are all smaller than 6. The following lemma will be used to handle odd entries of size greater than 6.

**Lemma 3.15.** *The nilpotent orbit $\mathcal{O}[7] \subseteq \mathfrak{o}_7$ is contained in $3C$.*

In the calculation proving this lemma, as in the rest of this subsection, we compute with concrete matrices that are skew-symmetric with respect to the bilinear form $(x, y) := \sum_i x_i y_{n+1-i}$.

*Proof.* A straightforward computation shows that the difference

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{2} & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 1 \\ 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is semisimple with three double eigenvalues $0, \frac{i}{2}$, and $-\frac{i}{2}$, so that it lies in $2C$ by the proof of Proposition 3.7. As the matrix on the right lies in $\mathcal{O}[7]$ and the matrix of the left lies in $C$, this proves the lemma. □

Recall from the introduction the notation $l(\mathbf{d})$ for the number of odd entries of $\mathbf{d}$ that are greater than 1. We are now in a position to prove Theorem 1.6.

*Proof of Theorem 1.6.* We prove the statement for partitions $[r, r]$ (with $r > 1$), $[2r + 1, 2s + 1]$ (with $r > s > 0$) and $[2r + 1]$ (with $r > 2$). The result will then follow by pairing equal even entries, pairing odd entries $> 1$, taking $[d_1]$ if $l$ is odd, and forming appropriate block matrices.

The orbit $\mathcal{O}[r, r]$ is represented by a $2r \times 2r$-matrix of the following form (drawn here for $r = 3$):

$$\left[\begin{array}{ccc|ccc} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & & & \\ \hline & & & 0 & -1 & \\ & & & & 0 & -1 \\ & & & & & 0 \end{array}\right].$$

This matrix is equal to $(E_{1,2} - E_{2r-1,2r}) + \cdots + (E_{r-1,r} - E_{r+1,r+2})$ (where $E_{ij}$ is the matrix with an entry 1 at $(i, j)$ and zeroes elsewhere). These $r - 1$ matrices all belong to $C$ (cf. Corollary 3.4), hence $\mathcal{O}[r, r] \in (r - 1)C$.

Next we consider the orbit $\mathcal{O}[2r+1, 2s+1]$ with $r > s > 0$. There is a simple recipe for finding a representative of this orbit [7, Recipe 5.2.4]; by way of example, the orbit $\mathcal{O}[7,3]$ is represented by (leaving out the off-diagonal zeroes):

$$\left[\begin{array}{ccccc|ccccc}
0 & 1 & & & & & & & & \\
 & 0 & 1 & 1 & & & & & & \\
 & & 0 & & 1 & & & & & \\
 & & & 0 & 1 & 1 & & & & \\
 & & & & 0 & & -1 & & & \\
\hline
 & & & & & 0 & -1 & -1 & & \\
 & & & & & & 0 & & -1 & \\
 & & & & & & & 0 & -1 & \\
 & & & & & & & & 0 & -1 \\
 & & & & & & & & & 0
\end{array}\right].$$

This matrix lies in $4C$, as it is the sum of $E_{1,2} - E_{9,10}, E_{2,3;4} - E_{7;8,9}, E_{3;4,5} - E_{6,7;8}$, and $E_{4,6} - E_{5,7}$, where we use the shorthand notation $E_{i,j_1;j_2}$ for $E_{i,j_1} + E_{i,j_2}$, and its analogue for rows. These matrices all belong to $C$ by Corollary 3.4, and a moment's reflection shows that this, too, generalises to the case where $r$ and $s$ are arbitrary, proving that $\mathcal{O}[2r+1, 2s+1] \subseteq (r+s)C$.

Finally, the orbit $\mathcal{O}[2r+1]$ with $r > 2$ has a representative of the form drawn in Lemma 3.15 for $r = 3$. Subtracting the $(r-3)$ matrices $E_{1,2} - E_{2r,2r+1}, E_{2,3} - E_{2r-1,2r}, \ldots, E_{r-3,r-2} - E_{r+4,r+5} \in C$ yields a matrix with zeroes everywhere except for the $7 \times 7$-block of Lemma 3.15 in the middle; that lemma shows that this matrix lies in $3C$. Hence, $\mathcal{O}[2r+1] \subseteq rC$.

To conclude the proof, consider first the case where $l(\mathbf{d})$ is even. We then partition the entries of $\mathbf{d}$ that are greater than 1 into pairs of the forms $[r,r]$ and $[2r+1, 2s+1]$ as above. In the case where $l(\mathbf{d})$ is odd and $d_1 > 5$, we decompose the entries $d_2, \ldots, d_m$ as in the first case, and form the singleton $[d_1]$. In both cases, a representative of $\mathcal{O}[\mathbf{d}]$ is then found by gluing the block matrices corresponding to the pairs, and in the second case the block matrix corresponding to the singleton $[d_1]$, together in an appropriate way. The above calculations show that $\mathcal{O}[\mathbf{d}]$ lies in $(\mathrm{rk}(A)/2)C$, as claimed. □

We conclude by recalling that, for some nilpotent orbits $\mathcal{O}$, half the rank of $\mathcal{O}$ does *not* suffice!

**Lemma 3.16.** *We have $\mathcal{O}[3, 1^{n-3}] \subseteq 2C \setminus C$ and $\mathcal{O}[3,2,2,1^{n-7}], \mathcal{O}[5, 1^{n-5}] \subseteq 3C \setminus 2C$.*

*Proof.* The first statement follows from Corollary 3.4: the matrix corresponding to the partition $[3, 1^{n-3}]$ has a non-zero square. It is clear that the orbits in the second statement are contained in $3C$. The claim follows then with Lemma 3.8, since both orbits have rank 4 and the square of a representative has rank 1 for the first, and 3 for the second orbit. □

The nilpotent orbits of smallest rank for which we do not know the smallest $k$ such that $kC$ contains them, are therefore $\mathcal{O}[3,3,3,1^{n-9}]$ and $\mathcal{O}[5,2,2,1^{n-9}]$, which are both of rank 6.

## 4. Conclusion and further research

We have successfully determined, for all classical simple Lie algebras $\mathfrak{g}$ and all $k \geq 1$, the sets $\overline{kC}$ where $C$ is the adjoint orbit of long root vectors, or, in the

terminology of [6], of extremal elements. If $\mathfrak{g}$ is $\mathfrak{sl}_n$ or $\mathfrak{sp}_{2n}$, then the sets $kC$ are closed, and the minimal $k$ for which they fill the whole space $\mathfrak{g}$ is equal to $n$ or $2n$, respectively. If, on the other hand, $\mathfrak{g}$ is $\mathfrak{o}_n$, then $2C$ is not closed, and we only know that the minimal $k$ for which $kC$ is equal to $\mathfrak{o}_n$ lies between $\lfloor \frac{n}{2} \rfloor$ and $\lfloor \frac{n}{2} \rfloor + 3$.

We conclude our paper with two rather speculative directions of further research, suggested by our findings. First, it is shown in [6] that the minimal number of elements of $C$ needed to *generate* $\mathfrak{g}$ *as a Lie algebra*, is equal to $n$ for $\mathfrak{sl}_n$, equal to $2n$ for $\mathfrak{sp}_{2n}$, and equal to $\lceil \frac{n}{2} \rceil$ for $\mathfrak{o}_n$. Of course, the similarity with the numbers that we listed above may be a coincidence, but if there should be a direct argument that these numbers are indeed equal, then the results of [6] could be used in solving the remaining open question concerning $\mathfrak{o}_n$, and in determining the secant varieties of the minimal orbit in the exceptional Lie algebras, as well.

As mentioned in the introduction, this paper is part of a rather ambitious project, namely: determining the higher secant varieties of the minimal orbit in arbitrary irreducible representations of reductive groups. In that setting, too, the complement of $kC$ in $\overline{kC}$ is worth investigation. The insight that, in the case of $\mathfrak{o}_n$, the complement of $2C$ in $\overline{2C}$ consists of a nilpotent orbit, suggests, in the general setting, that $\overline{kC} \setminus kC$ may be always contained in the null cone. However, if this were true, then it would follow from our Theorem 1.6 that $\lfloor \frac{n}{2} \rfloor C$ is already all of $\mathfrak{o}_n$, contrary to what a guess along the lines of the previous paragraph would yield.

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